increasing for $x \in (0, e)$ and it is decreasing for $x \in (e, +\infty)$. Since $\lim_{x \to 0^+} f(x) = -\infty$, and $f(e) = 1/e > \ln 2/2$ there is a unique root to the equation $\frac{\ln x}{x} = \frac{\ln 2}{2}$ in $(0, e)$, which is $x = 2$. Also, since $\lim_{x \to +\infty} f(x) = 0$, there is a unique root to the equation $\frac{\ln x}{x} = \frac{\ln 2}{2}$ in $(e, +\infty)$, which is $x = 4$. So, $x = 2$ and 4 are the only positive solutions to the problem.

Solution 2 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

For convenience we set $a = 2^{x+1}, b = x^2 + 2^x$ then (1) is equivalent to

$$a\left(1 - \sqrt{1+b}\right) = b\left(1 - \sqrt{1+a}\right) \Leftrightarrow \frac{ab}{1 + \sqrt{1+b}} = \frac{ab}{1 + \sqrt{1+a}}.$$

Since $ab \neq 0$, we get

$$\begin{aligned} \frac{1}{1+\sqrt{1+b}} = \frac{1}{1+\sqrt{1+a}} &\Longrightarrow 1 + \sqrt{1+b} = 1 + \sqrt{1+a} \\ &\Longrightarrow a = b \end{aligned}$$

Which means that $2^{x+1} = 2^x + x^2$ then 2^x taking ln of both sides we get

$$\frac{\ln x}{x} = \frac{\ln 2}{2}, \quad x > 1$$

Let $f : (1, \infty) \mapsto R$ be defined by $f(x) = \frac{\ln x}{x} - \frac{\ln 2}{2}$, then $f'(x) = x^{-2}(1 - \ln x)$ then f cannot have more than two roots (since f increases on $(1, e)$ and decreases on $(e, +\infty)$) and since $2, 4$ are obvious roots we conclude that the only positive solutions to the equation (1) are $2, 4$.

Also solved by Adnan Ali (student in A.E.C.S-4), Mumbai, India; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu, (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5329:** *Proposed by Arkady Alt, San Jose, CA*

Find the smallest value of $\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2}$ where real $x, y, z > 0$ and $xy + yz + zx = 1$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Since

$$\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= & \frac{1}{2}\left(\left(2x-y+\frac{y(x-y)^2}{x^2+y^2}\right)+\left(2y-z+\frac{z(y-z)^2}{y^2+z^2}\right)+\left(2z-x+\frac{x(z-x)^2}{z^2+x^2}\right)\right) \\
\geq & \frac{1}{2}\left((2x-y)+(2y-z)+(2z-x)\right) \\
= & \frac{x+y+z}{2} \\
= & \frac{1}{2\sqrt{2}}\sqrt{6(xy+yz+zx)+(x-y)^2+(y-z)^2+(z-x)^2} \\
= & \frac{1}{2\sqrt{2}}\sqrt{6} \\
= & \frac{\sqrt{3}}{2},
\end{aligned}$$

and equality holds when $x=y=z=\dfrac{1}{\sqrt{3}}$, so the smallest value required is $\dfrac{\sqrt{3}}{2}$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since real $x,y,z>0$ and $xy+yz+zx=1$, there is an acute triangle ABC such that $\cot A=x, \cot B=y$ and $\cot C=z$ so

$$\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= & \cot A-\frac{\cot A\cot^2 B}{\cot^2 A+\cot^2 B}+\cot B-\frac{\cot B\cot^2 C}{\cot^2 B+\cot^2 C}+\cot C-\frac{\cot C\cot^2 A}{\cot^2 C+\cot^2 A} \\
\geq & \cot A+\cot B+\cot C-\frac{\cot A\cot^2 B}{2\cot A\cot B}-\frac{\cot B\cot^2 C}{2\cot B\cot C}-\frac{\cot C\cot^2 A}{2\cot C\cot A} \\
= & \frac{1}{2}\left(\cot A+\cot B+\cot C\right) \\
\geq & \frac{\sqrt{3}}{2}
\end{aligned}$$

with equality iff $\cot A=\cot B=\cot C$ and $A=B=C=\pi/3$, that is iff
$x=y=z=\dfrac{1}{\sqrt{3}}$,

where we have used that $\cot A, \cot B > 0, (\cot A - \cot B)^2 \geq 0$ with equality iff $\cot A = \cot B$ and cyclically, and inequality 2.38 page 28, *Geometric Inequalities,* Bottema O., Djordjević, R.Ž., Janić, R.R., Mitrinović, D.S. Vasić, P.M., Wolters-Noordhoff, , Groningen, 1969.

Solution 3 by Henry Ricardo, New York Math Circle, NY

The AGM inequality gives us

$$\frac{x^3}{x^2+y^2} = x - \frac{xy^2}{x^2+y^2} \geq x - \frac{xy^2}{2xy} = x - \frac{y}{2}.$$

Similarly, we get

$$\frac{y^3}{y^2+z^2} \geq y - \frac{z}{2} \quad \text{and} \quad \frac{z^3}{z^2+x^2} \geq z - \frac{x}{2}.$$

Adding these three inequalities, we see that

$$f(x,y,z) = \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{x+y+z}{2}. \qquad (A)$$

Now we have

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = x^2+y^2+z^2+2,$$

so $x+y+z = \sqrt{x^2+y^2+z^2+2} \geq \sqrt{3}$, where we have used the well-known inequality $x^2+y^2+z^2 \geq xy+yz+zx$.

Thus $f(x,y,z) \geq \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

Editor′s comment : The author also provided a second solution to the above problem. It starts off exactly as the one above up until statement A. Then:

$$\begin{aligned} f(x,y,z) &= \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{x+y+z}{2} = \frac{3}{2}\left(\frac{x+y+z}{3}\right) \\ &\geq \frac{3}{2}\left(\frac{xy+yz+zx}{3}\right)^{1/2} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus $f(x,y,z) \geq \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

Solution 4 by Albert Stadler, Herrliberg, Switzerland

Suppose that $xy+y+zx=1$. We claim that

$$\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{\sqrt{3}}{2}, \qquad (1)$$

with equality if and only if $x=y=z=\dfrac{1}{\sqrt{3}}$. By homogeneity, (1) is equivalent to the unconditional inequality

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{\sqrt{3}}{2}\right). \qquad (2)$$

We first note that

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx \geq 3\left(xy+yz+zx\right),$$

since by the Cauchy-Schwarz Inequality, $x^2+y^2+z^2 \geq xy+yz+zx$, with equality if and only if $x=y=z$.

So

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{\sqrt{3}}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right).$$

To prove (2) it is therefore enough to prove that

$$\frac{1}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{1}{2} \qquad (3)$$

with equality if and only if $x=y=z$.

Clearing denominators we see that (3) is equivalent to

$$\sum_{cycl} x^5y^2 + \sum_{cycl} x^2y^5 + \sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4 + \sum_{cycl} x^4y^2z + \sum_{cycl} x^4yz^2. \qquad (4)$$

By the weighted AM-GM inequality,

$$\frac{1}{2}x^2y^2 + \frac{1}{2}x^4y^3 \geq x^3y^4,$$

$$\frac{3}{19}x^2y^5 + \frac{2}{19}y^3z^5 + \frac{14}{19}z^2x^5 \geq x^4yz^2,$$

$$\frac{1}{2}x^5y^2 + \frac{1}{2}x^3z^4 \geq x^4y^{z2},$$

$$\frac{10}{19}x^5y^5 + \frac{3}{76}y^5z^2 + \frac{7}{38}z^5x^2 + \frac{1}{4}x^4y^3 \geq x^4y^2z.$$

We conclude that

$$\frac{1}{2}\sum_{cycl} x^2y^5 + \frac{1}{2}\sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4, \qquad (5)$$

$$\frac{1}{2}\sum_{cycl} x^2y^5 = \frac{1}{2}\left(\frac{3}{19}\sum_{cycl} x^2y^5 + \frac{2}{19}\sum_{cycl} y^2z^5 + \frac{14}{19}\sum_{cycl} z^2x^5\right) \geq \frac{1}{2}\sum_{cycl} x^4yz^2, \qquad (6)$$

$$\frac{1}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{1}{4}\sum_{cycl} x^5y^2 + 14\sum_{cycl} x^3z^4 \geq \frac{1}{2}\sum_{cycl} x^4yz^3, \qquad (7)$$

$$\frac{3}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{10}{19}\sum_{cycl} x^5y^2 + \frac{3}{76}\sum_{cycl} x^5z^2 + \frac{7}{38}\sum_{cycl} z^5x^2 + \frac{1}{4}\sum_{cycl} x^4y^3 \geq 4\sum_{cycl} x^4y^2z. \qquad (8)$$

Condition (4) follows by adding (5),(6),(7), and (8). Equality holds if and only if $x = y = z$. (This is the equality condition for weighted AM-GM inequalities.)

Also solved by Adnan Ali (student, in A.E.C.S-4), Mumbai, India; Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Nicusor Zlota (plus a generalization) "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5330:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $B(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$ and let $n \geq 2$ be an integer.

Calculate the matrix product

$$B(2)B(3)\cdots B(n).$$

Solution 1 by Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania

We denote A(n)=B(1)B(2)...B(n). We have

$$A(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A(2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

We assume that

$$A(n) = \begin{pmatrix} \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \\ \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \end{pmatrix}. \qquad (1)$$

Since

$$A(n) = \begin{pmatrix} \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \\ \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \end{pmatrix}\begin{pmatrix} n+1 & 1 \\ 1 & n+1 \end{pmatrix} = \begin{pmatrix} \frac{(n+2)!}{2} & \frac{(n+2)!}{2} \\ \frac{(n+2)!}{2} & \frac{(n+2)!}{2} \end{pmatrix},$$

we have shown, by mathematical induction that (1) holds for all integers $n \geq 2$.

Solution 2 by Moti Levy, Rehovot, Israel

Let $B(x) = xI + A$, where A is an involute matrix (i.e., $A^2 = I$).
Let $P_n = B(2)\,B(3)\cdots B(n)$.